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Seiberg-Witten Solution from Matrix Theory

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Abstract

As another evidence for the matrix Discrete Light Cone formulation of M theory, we show how general integrable Hamiltonian systems emerge from BPS bound states of k longitudinal fivebranes. Such configurations preserve eight supercharges and by chain of dualities can be related to the solution of $\mathcal{N} = 2$ four-dimensional gauge theories. Underlying Hitchin systems on the bare spectral curve with k singular points arise from the Matrix theory compactification on the dual curve.

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Introduction

Being a candidate for the nonperturbative formulation of string theory [1], M theory plays a prominent role in the web of dualities and understanding of nonperturbative dynamics in various vacua. One of its notable applications is the solution of $\mathcal{N} = 2$ SUSY gauge theories in four dimensions [2]. The Coulomb branch of the corresponding field theory comes as an effective theory of a background brane configuration in string theory. From the construction it is clear that integrability is not inherent to four-dimensional physics, but seems to be a characteristic feature of supersymmetric BPS state in string theory. In fact, a spectral curve in [2] appears as a supersymmetric cycle which is a part of M-fivebrane world-volume. Because the coupling constant of the effective four-dimensional theory does not depend on the string coupling constant, nonperturbative dynamics in $D = 4$ may be extracted by studying classical configurations in string theory.

There was no explicit formulation of M theory as a quantum theory before the conjecture of [3]. In that paper it was proposed to refer to $U(N)$ matrix quantum mechanics of D-particles in the $N \rightarrow \infty$ limit as exact description of M theory in the infinite momentum frame. Then branes [3, 4, 5], strings [6, 7] and black holes [8, 9] were realized in terms of the Matrix model; their scattering was shown as well to match corresponding string theory amplitudes in many recent works. The matrix formulation of M theory has already passed a number of tests via compactification to various dimensions [10] and in this line the Seiberg-Witten exact solution [11] could be another challenging consistency check for the Matrix model.

In the present work we make the first step in this direction ¹. Surprisingly, the Matrix theory gives the clear physical answer to the question: "Why integrability" from the first principles. To begin with, let us emphasize what we exactly mean by *Matrix theory*. In order to accord to $SU(N)$ gauge theory the solution require finite number N of D0-brane charge. This is very similar to the compactification of the Matrix theory on a three-torus which yields $\mathcal{N} = 4$ Yang-Mills theory in $D = 4$ that possesses exact S-duality group. This

¹Related efforts were also made in [12, 13]

and other arguments were accumulated in [14] to put the proposal [3] further and to define finite N formulation of the Matrix theory in terms of the Discrete Light Cone Quantization (DLCQ). Therefore, investigating finite N dynamics, we give another evidence for the DLCQ approach.

As was already mentioned above, supersymmetry is another vital ingredient. In the next section we show that supersymmetry arguments in the Matrix model restrict our attention to longitudinal fivebrane backgrounds. In type IIA theory it gives rise to bound state of D4-branes embedded in a four-dimensional part \mathcal{M} of transverse space, D0-branes and two mutually orthogonal stacks of D2-branes also in \mathcal{M} . The solution is then defined by topological data only, e.g. genus of a spectral curve which fixes the rank of gauge group and so on. Namely, in our case a particular integrable system is selected out by brane charges and manifold \mathcal{M} .

Taking elliptic models as examples of Hitchin system [15, 16], in section 2 we investigate toroidal compactification of the Matrix theory to 9 dimensions. This theory is equivalent to type II string theory [6, 10], and the longitudinal fivebrane background on \mathcal{M} implies no D2-brane charges because of noncompact form of $\mathcal{M} = R^2 \times T^2$ and finite N . The bound state carry N units of D0-brane charge and k units of D4-brane charge. The fourbrane charges correspond to k (dimensionally reduced) "instantons" in the gauge theory on the dual manifold \mathcal{W} . The basic idea is that manifold \mathcal{W} does not include time variable of the SYM_{2+1} theory relevant to the Matrix theory compactification. The "instantons" on \mathcal{W} are particles in $U(N)$ gauge theory on $R \times \mathcal{W} = (\text{time}) \times \mathcal{W}$. Therefore these "*instantons*" on \mathcal{W} are *point-like states in $(2 + 1)$ -dimensional Super-Yang-Mills (SYM_{2+1}) theory on $R \times T^2$* . We will show that incorporating them into SYM_{2+1} theory results in integrable Hitchin system on the torus with k punctures [17, 18]. Moreover, in the framework of the Matrix theory all integrable systems and their spin generalizations come on the same footing, reasserting the mightiness of the Matrix theory. In the double scaling limit bare torus degenerates into a sphere recovering the Seiberg-Witten exact solution.

Section 3 is devoted to string theory interpretation of the results of section 2 and their relation to the solution of $\mathcal{N} = 2$ gauge theories in four dimensions via M-fivebrane [2]. This

clarifies the analogy between punctures on the bare spectral curve and NS5-branes in brane configurations.

General properties of the longitudinal matrix fivebrane on arbitrary \mathcal{M} in the form $\mathcal{M} = \mathcal{C} \times \mathcal{S}$ and algebraically completely integrable Hamiltonian systems behind such compactifications are discussed in section 4.

We conclude in section 5 by some remarks on the rational models, i.e. those relevant to theories with fundamental matter.

1 The Power of Supersymmetry

Ten-dimensional $U(N)$ Super-Yang-Mills theory reduced to $0 + 1$ dimensions has been proposed to describe M theory in the Discrete N Light Cone formalism [3, 14]. Eleven-dimensional action written in terms of $N \times N$ Hermitean matrices X and Ψ :

$$S = \text{Tr} \int dt \left(\frac{1}{2R} (D_0 X^i)^2 - \frac{1}{4} R [X^i, X^j]^2 - \bar{\Psi} D_0 \Psi - R \bar{\Psi} \Gamma^i [X_i, \Psi] \right) \quad (1.1)$$

depends on the Plank length l_p and the radius of the compact eleventh direction R .

According to [3], compactification of the original M theory on some manifold \mathcal{M} opens up extra dimensions in Yang-Mills theory on the manifold \mathcal{W} of the inverse size ². For example, L_1, \dots, L_d toroidal compactification on $\mathcal{M} = T^d$ leads to the $(d+1)$ -dimensional Yang-Mills theory on the dual torus $\mathcal{W} = \tilde{T}^d$ with sides ³:

$$\Sigma_i = \frac{l_p^3}{R L_i} \quad (1.2)$$

SYM _{$d+1$} coupling constant [10]

$$g_{YM}^2 = \frac{R^3}{l_p^6} \prod_i \left(\frac{l_p^3}{R L_i} \right) \quad (1.3)$$

goes to infinity when the compact manifold \mathcal{M} shrinks to zero size. In this limit one has to replace $N \times N$ matrices X by covariant derivatives:

$$X_\mu \rightarrow -i D_\mu = -i \left(\frac{\partial}{\partial x_\mu} + A_\mu \right) \quad (1.4)$$

² \mathcal{M} does not include the 11th direction which will be assumed large for the rest of the paper.

³We omit factors of 2π .

with respect to $G = U(N)$ -valued gauge connection A_μ , so that the corresponding action takes the following form:

$$S = \frac{1}{g_{YM}^2} \text{Tr} \int_{R \times \mathcal{M}} dt d^d x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \bar{\Psi} \Gamma^\mu [X_\mu, \Psi] \right) \quad (1.5)$$

where $F_{\mu\nu} = [X_\mu, X_\nu]$ with the appropriate replacement (1.4) of compact X_μ . Here and in the following the indices μ, ν run from 0 to 9.

Action (1.5) enjoys 16 dynamical SUSY transformations:

$$\begin{aligned} \delta X_\mu &= i\bar{\epsilon} \Gamma_\mu \Psi \quad \mu = 0, 1, \dots, 9 \\ \delta \Psi &= (D_0 X_i) \Gamma^{0i} \epsilon + \frac{i}{2} [X_i, X_j] \Gamma^{ij} \epsilon \quad i, j = 1, 2, \dots, 9 \end{aligned} \quad (1.6)$$

and 16 kinematical supersymmetries:

$$\delta X_\mu = 0 \quad \delta \Psi = \tilde{\epsilon} \quad (1.7)$$

which have nonlinear realization and are never preserved by themselves. For this reason in the sequel we will search for BPS states preserving half of the dynamical SUSY only, i.e. eight real supercharges.

Antisymmetric in μ, ν matrix $F_{\mu\nu}$ can always be brought to Jordan form:

$$F_{\mu\nu} = \begin{pmatrix} 0 & F_{01} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -F_{01} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -F_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & F_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -F_{45} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{67} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -F_{67} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{89} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -F_{89} & 0 \end{pmatrix} \quad (1.8)$$

The matrix F corresponding to Dp-brane BPS state preserving $\frac{1}{2^{\lfloor \frac{p}{2}-1 \rfloor}}$ fraction of the dynamical SUSY (1.6) satisfies the following conditions [19]:

$$F_{12} - \sum_{i=2}^{\frac{p}{2}} \xi_i F_{(2i-1)(2i)} = 0 \quad \Gamma_{12} \Gamma^{(2i-1)(2i)} \epsilon = \xi_i \epsilon \quad (1.9)$$

Therefore, eight real supercharges are preserved for a BPS background with, say $F_{45} = F_{67}$ and all the other fields in (1.5) put to zero. This background corresponds to the marginal bound state of a $D(p=4)$ -brane with N D0-branes and two equal sets of D2-branes [19, 20]. The total Dp-brane topological charge of the field configuration is [4, 5]:

$$Q_p \propto \int \text{Tr} F^{\wedge(\frac{p}{2})} = \int \text{Tr} F \wedge \dots \wedge F \quad (1.10)$$

Note that like in [2] supersymmetry arguments lead us to the longitudinal fivebrane background, but now in the very different way.

Because all the background fields in the directions other than x^4, x^5, x^6 and x^7 must vanish, we consider k longitudinal fivebranes on a four-manifold $\mathcal{M} = (x^4, x^5, x^6, x^7)$. From the Yang-Mills side it means charge k instanton solution in $U(N)$ gauge theory on \mathcal{W} [20]:

$$k = \frac{1}{8\pi^2} \int \text{Tr} F \wedge F \quad (1.11)$$

which has a moduli space of self-dual (or anti-self-dual for negative k) field strengths $\tilde{F} = \pm F$ ⁴.

$$\begin{cases} F_{45} = [X_4, X_5] = [X_6, X_7] = F_{67} \\ F_{57} = [X_5, X_7] = [X_6, X_4] = F_{64} \\ F_{74} = [X_7, X_4] = [X_6, X_5] = F_{65} \end{cases} \quad (1.12)$$

Self-duality of the field F on \mathcal{W} is the key point in our construction and we will make use of it in the spirit of [16] and [32]. The crux of the matter is that from the Matrix theory compactification on \mathcal{M} we get $(4+1)$ -dimensional $U(N)$ gauge theory on $R \times \mathcal{W} = (\text{time}) \times \mathcal{W}$ with k four-dimensional "instantons" on \mathcal{M} which does not include time! From the SYM_{4+1} point of view *these "instantons" look like k particles on $R \times \mathcal{W}$* . We will use this trick in the next section to make a reduction with respect to a subgroup and impose a moment map condition. The important role of the self-dual field was also discussed in [21] for the equivalent (by T-duality in x^6, x^7 directions) configuration of two-branes intersecting at angles.

Let us stress here the essential role of supersymmetry. Indeed, all the arguments we made so far were based on preserving a quarter of SUSY in the Matrix theory. In the next

⁴We always imply covariant derivative (1.4) for compact X .

sections we will show that specifying the manifold \mathcal{M} we determine the system completely which has a natural integrable structure in the holomorphic sense [15]. In what follows we will consider only compactifications of the Matrix theory on a direct product of two complex surfaces $\mathcal{M} = \mathcal{C} \times \mathcal{S}$ parametrized by (local) holomorphic coordinates:

$$w = x^4 + ix^5 \in \mathcal{C} \quad \text{and} \quad z = x^6 + ix^7 \in \mathcal{S} \quad (1.13)$$

and let us take $\mathcal{C} = R^2$ and $\mathcal{S} = T^2$ as the first example.

2 Elliptic Models

Elliptic integrable models usually arise in $(2+1)$ - and $(1+1)$ -dimensional topological gauge theories on the same torus [24, 25, 26, 30]. This is a strong motivation to study toroidal compactification of the Matrix theory which gives rise to Super-Yang-Mills in $2+1$ dimensions [10, 23]. By this reason in the present section we examine a compactification of k longitudinal fivebranes on $\mathcal{M} = \mathcal{C} \times \mathcal{S} = R^2 \times T^2$, where the torus:

$$\begin{aligned} y^2 &= (x - e_1)(x - e_2)(x - e_3) \\ x &= \wp(z) \quad y = \wp'(z) \quad dz = \frac{dx}{y} \end{aligned} \quad (2.1)$$

has sides L_6 and L_7 , and $\wp(z)$ is the double periodic Weierstrass \wp -function with periods 1 and $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$. SYM_{2+1} theory is defined on the dual torus $\tilde{\mathcal{S}}$ with the same complex structure but with the inverse sides $\Sigma_6 = \frac{l_p^3}{RL_6}$ and $\Sigma_7 = \frac{l_p^3}{RL_7}$.

In these notations self-duality equations (1.12) for $F_{\mu\nu}$ take the form:

$$\begin{aligned} F &= \frac{i}{2} [\phi, \bar{\phi}] \\ \bar{\mathcal{D}}\phi &= 0 \end{aligned} \quad (2.2)$$

where $\mathcal{D} = D_6 + iD_7$, $F = \frac{i}{2} [\mathcal{D}, \bar{\mathcal{D}}]$, $\phi = X^4 + iX^5$ and the gauge connection $A = A^6 + iA^7$ take values in the complexified Lie algebra $G_C = U_C(N)$ for the case at hand.

Remarkably, the second equation in (2.2) is nothing but a Gauss law in the $(2+1)$ -dimensional gauge theory on $R \times \tilde{\mathcal{S}}$. This can be easily seen by introducing a source term

in the action (1.5):

$$\delta S_{\text{source}} = \text{Tr} \int_{R \times T^2} dt d^2 z A_0 J_0 \delta(z) \quad (2.3)$$

Concentrating on holomorphic bundles over the torus, the relevant part of the action (1.5) looks like [30]:

$$\begin{aligned} S &= \frac{1}{g_{YM}^2} \text{Tr} \int_{R \times T^2} dt d^2 z \left[-F_{0\bar{z}} \Phi + \frac{1}{2} \Phi^2 + A_0 J_0 \delta(z) + \dots \right] = \\ &= \frac{1}{g_{YM}^2} \text{Tr} \int_{R \times T^2} dt d^2 z \left[A_0 \left(-\bar{\mathcal{D}} \Phi + J_0 \delta(z) \right) + \frac{1}{2} \Phi^2 + \dots \right] \end{aligned} \quad (2.4)$$

where we have introduced an auxiliary field $\Phi = F_{0\bar{z}}$.

But one shall not forget about k "instantons" on \mathcal{W} or, equivalently, about k particles on $R \times \mathcal{W}$. The "instantons" we discuss here resemble actual four-dimensional instantons only when \mathcal{M} is compact, i.e. \mathcal{W} is four-dimensional. In the elliptic case of this section an "instanton" means a self-dual solution to the equations (1.12) with non-zero charge (1.11) in two dimensions. For this reason we distinguish it by quotation-marks from the four-dimensional instanton.

Therefore, implementing k "instantons" means taking k source terms (2.3) that modify the Gauss law (2.2) to [18, 22, 31]:

$$\bar{\partial} \Phi_{ij} + (a_i - a_j) \Phi_{ij} = \sum_{\alpha=1}^k J_{ij}^{(\alpha)} \delta(z - z_\alpha) \quad (2.5)$$

where we used the residual gauge symmetry on the torus to take A in the diagonal form $A = \text{diag}(a_1, \dots, a_N)$. Matrices $J_{ij}^{(\alpha)}$ determine the orientation of $(2+1)$ -dimensional particles in the color group. Physically, taking $A_0 = 0$ gauge, one must bear in mind the Gauss law (2.5) which turns out to be a moment map condition!

Let us outline the entire chain of points we followed to obtain the moment map equation (2.5) from the $D4 - D0$ bound state in the Matrix theory:

$$\begin{array}{c}
\text{Supersymmetric vacuum} \\
\downarrow \\
\text{Longitudinal fivebrane on } \mathcal{M} \\
\downarrow \\
\text{Self-dual field on } \mathcal{W} \\
\downarrow \\
U(N) \text{ "instanton" on } \mathcal{W} \\
\downarrow \\
\text{A particle in SYM}_{2+1} \text{ on } R \times \mathcal{W} \\
\downarrow \\
\text{The moment map condition}
\end{array} \tag{2.6}$$

In the language of integrable system each source (2.3) represents a marked point on the bare spectral curve $\tilde{\mathcal{S}}$ of the corresponding Hitchin system [17, 18, 22, 24]. It means that the number of longitudinal matrix fivebranes k is exactly the number of singular points on $\tilde{\mathcal{S}}$. Moreover, in the Matrix program integrable data appear in a clear physical way, e.g. stable pair (V, Φ) of Hitchin system is defined by $G = U_c(N)$ vector bundle and the adjoint field Φ of $U(N)$ SYM₂₊₁ compactification.

Nontrivial solution to the moment map equation (2.5):

$$\Phi_{ij}(z) = \delta_{ij} \left(p_i + \sum_{\alpha} J_{ij}^{(\alpha)} \partial \log \theta(z - z_{\alpha} | \tau) \right) + (1 - \delta_{ij}) e^{a_{ij}(z - \bar{z})} \sum_{\alpha} J_{ij}^{(\alpha)} \frac{\theta(z - z_{\alpha} + \frac{Im\tau}{\pi} a_{ij}) \theta'(0)}{\theta(z - z_{\alpha}) \theta(\frac{Im\tau}{\pi} a_{ij})} \tag{2.7}$$

is nothing but the Lax operator for general elliptic Gaudin system [18], where we have denoted $a_{ij} = a_i - a_j$ for brevity. A gauge transformation:

$$\Phi_{ij} \rightarrow (U^{-1} \Phi U)_{ij}(z) \quad U_{ij} = e^{a_{ij} \bar{z}} \tag{2.8}$$

takes it to the holomorphic form:

$$\Phi_{ij}(z) = \delta_{ij} \left(p_i + \sum_{\alpha} J_{ij}^{(\alpha)} \partial \log \theta(z - z_{\alpha} | \tau) \right) + (1 - \delta_{ij}) e^{a_{ij} z} \sum_{\alpha} J_{ij}^{(\alpha)} \frac{\theta(z - z_{\alpha} + \frac{Im\tau}{\pi} a_{ij}) \theta'(0)}{\theta(z - z_{\alpha}) \theta(\frac{Im\tau}{\pi} a_{ij})} \tag{2.9}$$

Introducing the spectral curve:

$$P(\lambda, z) = \det_{N \times N}(\lambda \delta_{ij} - \Phi_{ij}(z)) = 0 \quad (2.10)$$

and the meromorphic 1-differential dS on it:

$$dS = \lambda dz \quad \frac{\partial dS}{\partial h_j} = \text{holomorphic differential} \quad (2.11)$$

one can easily find the solutions of string vacua or $\mathcal{N} = 2$ field theories. For instance, masses of BPS multiplets are defined by periods of dS :

$$A_i = \int_{\alpha_i} dS \quad A_i^D = \int_{\beta_i} dS \quad (2.12)$$

over the basic cycles $\alpha_i \circ \beta_j = \delta_{ij}$ on the spectral surface (2.10).

Somewhat miraculously, unlike (1.11), two-brane charge $\int \text{Tr} F = \int \text{Tr} [X^4, X^5]$ vanishes identically because of two noncompact directions x^4 and x^5 and finite N . Hence the extra constraint on matrices $J_{ij}^{(\alpha)}$ [18]:

$$\sum_{\alpha=1}^k J_{ii}^{(\alpha)} = 0 \quad (2.13)$$

is automatically satisfied in the realm of the Matrix theory. The absence of D2-branes in the system will also prove essential in the next section where we study the relation of this system to the brane construction of [2]. If there were nonzero D2-brane charges, it would be not possible to map the matrix brane configuration to that of [2] where it was used to derive the solutions of $\mathcal{N} = 2$ gauge theories in four dimensions. Present approach naturally generalizes elliptic models of [2] to spin degrees of freedom. Namely, the solution (2.9) in the most general form corresponds to the elliptic Gaudin model.

For the sake of simplicity, let us take an illustrative example of elliptic Calogero-Moser model. This system describes N pair-wise interacting particles on the torus with the Hamiltonian:

$$H = h_2 = \sum_i \frac{p_i^2}{2} + \frac{m^2}{8} \sum_{i < j} \wp(x_i - x_j) \quad (2.14)$$

Lax operator (2.9) for this model follows from the solution of the moment map equation (2.5) with the only marked point where the residue is $J_{ij} = m(1 - \delta_{ij})$ which means taking a brane background with N D0-branes and $k = 1$.

In the double-scaling limit

$$m^2 \rightarrow \infty \quad q = e^{i\pi\tau} \rightarrow 0 \quad (2.15)$$

so that $\Lambda^N \propto m^N q$ remains finite, the system goes to the periodic Toda chain which is behind the integrability of $\mathcal{N} = 2$ supersymmetric $SU(N)$ pure gauge theory in four dimensions [27]. In the brane construction [2] it also occurs as degeneration of the elliptic model that gives a hint for the equivalence of the bare spectral curve (2.1) and space-time torus in [2]. The analogy will be corroborated in the next section.

N -periodic Toda chain describes one-dimensional system of N particles on a circle with exponential interaction of the nearest neighbours. In this limit the spectral curve (2.10) degenerates into double covering of sphere and takes a very simple form:

$$\Lambda^N \cosh(z) = 2P_{(N)}(\lambda) \quad (2.16)$$

that coincide with that of the Seiberg-Witten solution of $\mathcal{N} = 2$ four-dimensional gauge theory [11, 27].

Another reward of the Matrix theory is that the prepotential and vacuum expectation values of $\mathcal{N} = 2$ gauge theory also come into the story. Thus, order parameters on the Coulomb phase are functionally independent, Poisson-commuting Hamiltonians of the integrable dynamics [24, 27]:

$$h_j = \frac{1}{j} \langle \phi^j \rangle \quad h_j \approx \frac{1}{j} \sum_{i=1}^N p_i^j + \dots \quad \langle \phi^j \rangle \approx \sum_{i=1}^N A_i^j + \dots \quad (2.17)$$

The number of Hamiltonians h_j is the same as the genus of the spectral curve (2.10) $g = Nk - \frac{k(k+1)}{2} + 1$ from the Riemann-Roch theorem.

The homogeneity of differential (2.11) allows one to express the prepotential \mathcal{F} via Hamiltonians (2.17). For the particular choice of $SU(2)$ gauge group, \mathcal{F} is a solution to the equation [28]:

$$a \frac{\partial \mathcal{F}}{\partial a} - 2\mathcal{F} = \frac{2i}{\pi} H \quad (2.18)$$

3 M-branes versus Matrix branes

In this section we prove that the brane configuration of [2] corresponding to the $\mathcal{N} = 2$ four-dimensional SYM theory with adjoint hypermultiplet represents the same background as the Matrix theory setup. For this purpose let us briefly remind the brane picture [2].

Classically we take N D4-branes with x^0, x^1, x^2, x^3, x^6 world-volume coordinates to get $U(N)$ Yang-Mills gauge theory with 16 real supercharges. In order to have $\mathcal{N} = 4$ four-dimensional low-energy field theory, one has to compactify x^6 direction on a circle of radius $Y_6 = \frac{g_{st} l_s}{g^2}$. The fourbranes are all at the same position along the x^7, x^8, x^9 directions and have different $v = x^4 + ix^5$ coordinates. To break supersymmetry further by a half one introduces a fivebrane with the world-volume in $x^0, x^1, x^2, x^3, x^4, x^5$. The space-time metric allows a nontrivial \mathbf{C} -bundle over S^1 :

$$x^6 \rightarrow x^6 + Y_6 \quad v \rightarrow v + m \quad x^{11} \rightarrow x^{11} + Y_{11} \theta \quad (3.1)$$

with arbitrary parameter m corresponding to the bare mass of the adjoint $\mathcal{N} = 2$ hypermultiplet that softly breaks $\mathcal{N} = 4$ supersymmetry. Varying m we interpolate between $\mathcal{N} = 4$ $SU(N)$ SYM theory ($m = 0$) and pure gauge $\mathcal{N} = 2$ theory (in the double scaling limit (2.15)). Now let us turn to the Matrix picture.

As it was explained in the previous section, type IIA theory description of theories with 8 real supercharges comes from the $4 - 2 - 2 - 0$ bound state. We choose the world-volumes of the branes to span x^4, x^5, x^6, x^7 for the D4-branes, x^4 and x^5 for the first set of D2-branes, x^6 and x^7 for the second set of D2-branes and D0 branes are allowed to move in all of the x^4, x^5, x^6, x^7 directions. If we restrict ourselves to the case of $U(N)$ gauge theory with adjoint matter, we have to consider the only D4-brane with N units of D0-brane charge without two-branes. In the language of the Matrix theory it means taking a longitudinal fivebrane in the DLCQ formulation of $U(N)$ matrix model at finite N . Moreover, for this particular case (of elliptic Calogero system) x^6 and x^7 must compound the bare torus (2.1) with modular parameter τ .

The equivalence of the two brane configurations comes from the chain of dualities under which a fivebrane goes into matrix longitudinal fivebrane, and each D4-brane gets mapped

into a D0-brane. If we start from the matrix picture of $D4 - D0$ bound state:

$$\left\{ \begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ D4 & + & - & - & - & + & + & + & + & - & - \\ D0 & + & - & - & - & - & - & - & - & - & - \end{array} \right. \quad (3.2)$$

the sequence is the following:

- T-duality along x^6 and x^7

It takes the T^2 to the dual torus with inverse size according to (1.2), but does not change its complex structure. Therefore, sides of the torus become Σ_6 and Σ_7 of the Yang-Mills theory. And the brane configuration changes to:

$$\left\{ \begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ D2 & + & - & - & - & + & + & - & - & - & - \\ D2 & + & - & - & - & - & - & + & + & - & - \end{array} \right. \quad (3.3)$$

where a "plus" stands for the extended direction of a brane and a "minus" corresponds to its fixed position.

- T-duality along x^1, x^2, x^3

This duality makes two sets of D5-branes in type IIB string theory with four common directions x^0, x^1, x^2 and x^3 which will be the space-time of the effective four-dimensional theory:

$$\left\{ \begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ D5 & + & + & + & + & + & + & - & - & - & - \\ D5 & + & + & + & + & - & - & + & + & - & - \end{array} \right. \quad (3.4)$$

Finally, to get the brane configuration of [2], we recall the eleventh dimension and perform

- $7 \leftrightarrow 11$ Flip

It does not change anything from the M theory point of view because of the full 11-dimensional Lorentz invariance (but, of course, modifies the coupling constant g_{st} of type IIA string theory). We come to the single fivebrane:

$$\left\{ \begin{array}{c|cccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 11 \\ NS5 & + & + & + & + & + & + & - & - & - & - & - \\ D4 & + & + & + & + & - & - & + & - & - & - & + \end{array} \right. \quad (3.5)$$

that has $R^4 \times \Sigma$ world-volume, where Σ is the spectral curve (2.10) of the underlying integrable system holomorphically embedded into $R^2 \times T^2$ where the torus $\tilde{\mathcal{S}}$ has periods 1 and τ and sides Σ_6 and Σ_7 . The classical picture of N $D4$ -branes suspended in the background of NS5-brane is restored in the type IIA limit when radius of the 11th dimension goes to zero.

We have established the perfect agreement between the Matrix solution via longitudinal fivebrane and the standard M theory approach [2]. One can apply this (or reverse) chain of dualities to relate any brane constructions of field theories in the Coulomb phase to their matrix counterparts.

4 General \mathcal{M}

In this section we make several remarks on the Matrix theory compactification on four-manifold \mathcal{M} in the most general form $\mathcal{M} = \mathcal{C} \times \mathcal{S}$. The most straightforward one is the extension of the analysis in the previous sections to arbitrary genus $g > 1$ compact Riemann surface $\tilde{\mathcal{S}}$ while \mathcal{C} is still a complex plane. This is exactly the subject of [15] where it was shown that even without singular points on $\tilde{\mathcal{S}}$ we end up with an algebraically completely integrable Hamiltonian system. In fact, holomorphic vector bundle V of rank N over the compact smooth Riemann surface $\tilde{\mathcal{S}}$ and its section Φ arise from $U(N)$ SYM₂₊₁ theory on $R \times \tilde{\mathcal{S}}$ in a very natural way. By the vanishing theorem [16], the pair (V, Φ) arising from a solution of the self-duality equations (2.2) is necessarily stable. Cotangent bundle T^*M_V of the moduli space M_V of this stable bundle V is Hitchin system, because the number of Poisson-commuting Hamiltonians h_j is exactly the same as the dimension of the moduli space $\dim(M_V) = N^2(g - 1) + 1$ [15, 17].

Another possible compactification is on $\mathcal{M} = \mathcal{C} \times \mathcal{S}$ where the size of the compact

manifold \mathcal{C} goes to infinity and fivebrane charge k is turned off. On the Yang-Mills side it looks like compactification of $\mathcal{N} = 4$ (instead of $\mathcal{N} = 2$) four-dimensional $U(N)$ Yang-Mills theory without "source" on $\mathcal{W} = \tilde{\mathcal{C}} \times \tilde{\mathcal{S}}$ where $V_{\tilde{\mathcal{C}}} \rightarrow 0$. Similar reduction of slightly different (twisted) $\mathcal{N} = 4$ theories was shown to yield supersymmetric 2D sigma-model on hyperKahler Hitchin space (namely, compactified cotangent bundle T^*M_V to the moduli space M_V of flat connections) [32].

Finally, one can consider compactification on four-manifold \mathcal{M} with a shrinking three-cycle in it. It results in $\mathcal{N} = 2$ supersymmetric four-dimensional $U(N)$ Yang-Mills theory in strong coupling regime. The same reasoning as in [10] explains that it probes infrared behaviour of the theory where we again come to the usual Seiberg-Witten story [11].

It seems possible to continue the list of integrable string vacua by introducing D2-branes and extending to various \mathcal{M} . The arguments of this section and detailed study of elliptic models in section 2 strongly suggest integrability of arbitrary vacuum in the Matrix theory preserving eight supercharges. We hope that this assumption will be illuminated in future. Note, that using the results of section 3 one can conjecture that the fivebrane of [2] lives on the *dual* space $\mathcal{C} \times \tilde{\mathcal{S}}$, i.e. spectral curve of the solution must be holomorphically embedded in $\mathcal{C} \times \tilde{\mathcal{S}}$.

5 Discussion on Rational Models

Applying the basic idea of section 4 to toroidal compactification of the longitudinal fivebrane in the DLCQ formulation of the Matrix theory, we derived in section 2 the elliptic Hamiltonian systems of Hitchin type. Had we been aware of integrability in the related brane background [2], we would have found it in the dynamics of $4 - 2 - 2 - 0$ bound state. This approach naturally extends known integrable string theory vacua to spin degrees of freedom of the integrable counterparts, but the original Seiberg-Witten solution [11] arise only in the double scaling limit (2.15) as the torus $\tilde{\mathcal{S}}$ degenerates into a sphere. This poses a problem how to obtain rational models also in a straightforward way. This is a very interesting subject for future work, because rational theories, i.e. $\mathcal{N} = 2$ four-dimensional gauge theories

with fundamental matter generally corresponding to spin magnets [29], are solvable due to the pivotal concept of R-matrix which is not manifest in the present approach.

Now we sketch an indirect argument for the equivalence of integrable inhomogeneous spin chains and certain compactifications of Heterotic Matrix theory [33, 34]. Completely different way to break supersymmetry (from what we have used in the main part of the text) is to study a membrane compactification on the orbifold $I \times S^1$. Corresponding $(2+1)$ -dimensional Yang-Mills theory defined on the dual manifold $S^1 \times \tilde{S} = I \times T^2$ has the desired SUSY.

The first sign of the Ruijsenaars system can be seen already at this stage from the D2-brane defining equation [3, 4, 5]:

$$UVU^{-1}V^{-1} = Z \quad (5.1)$$

This monodromy over the torus in the Chern-Simons theory gives rise [26] to the spectrum of the Ruijsenaars model.

The theory described above corresponds to compactification of Heterotic Matrix theory where supersymmetry and gauge symmetry anomaly cancellation arguments require to include a Chern-Simons term:

$$S_{CS} = \frac{\kappa}{2} \text{Tr} \int (AdA + \frac{2i}{3} A^3) \quad (5.2)$$

to the $(2+1)$ -dimensional Yang-Mills action [35, 34]. It modifies equations of motion of the Matrix model to [12]:

$$\frac{\partial^2 X^\mu}{\partial t^2} = -\kappa^2 X^\mu - \frac{3\kappa}{2} \epsilon^{\mu\nu\eta} [X^\nu, X^\eta] - [X^\nu [X^\mu, X^\nu]] \quad (5.3)$$

which have common solutions with the first-order equation:

$$\frac{\partial X^\mu}{\partial t} = -i\kappa X^\mu + \frac{i}{2} \epsilon^{\mu\nu\eta} [X^\nu, X^\eta] \quad (5.4)$$

They are nothing but the Nahm equations for $SU(2)$ monopole of charge N under the appropriate change of variables [36]:

$$\frac{\partial T_i}{\partial t} = \frac{1}{2} \epsilon^{ijk} [T_j, T_k] \quad (5.5)$$

The latter is equivalent to the inhomogeneous XXX $Sl(N)$ spin chain which governs the dynamics of $SU(2)^{N-1}$ gauge theory with fundamental matter [29].

The shortcut to this lengthy solution of models corresponding to theories with fundamental matter as well as other aspects of Heterotic Matrix theory compactification make up the subject of future investigation.

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